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An Alternate Proof of the Thompson Replacement Theorem

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Thompson's replacement theorem (see [1], page 273) is a result about p-groups which has received wide application in the study of arbitrary finite groups. The method of proof used by Thompson involves calculations with commutators of elements. The proof given here avoids these calculations and uses subgroups instead of elements. It also proves a somewhat more general result.

THEOREM. *Let G be a finite nilpotent group and let $A \subseteq G$ be Abelian. Suppose that $B \subseteq G$ is also Abelian, $A \subseteq \mathbf{N}(B)$ and $B \nsubseteq \mathbf{N}(A)$. Then there exists an Abelian subgroep $A^* \subseteq G$ such that*

(a) $|A| = |A^*|$,

(b) $A \cap B < A^* \cap B$,

(c) $A^* \subseteq \mathbf{N}(A)$

and (d) *if $|A|$ is odd, then the exponent of A^* divides that of A.*

Proof. We assume $G = AB$ and proceed by induction on $|G|$. Since $A \ntriangleleft G$, we have $A < G$ and thus $A \subseteq M$, a maximal subgroup of G. Since G is nilpotent, $M \triangleleft G$. If $B \cap M \nsubseteq \mathbf{N}(A)$, then by induction applied to M, there exists $A^* \subseteq M$ satisfying (a), (c), and (d) and such that $A \cap (B \cap M) < A^* \cap (B \cap M)$. Since $A, A^* \subseteq M$, (b) follows and we are done in this case.

We therefore assume that $B \cap M \subseteq \mathbf{N}(A)$ and hence $A \triangleleft A(B \cap M) = M$. For some $b \in B$, we have $b \notin \mathbf{N}(A)$ so that $A \neq A^b \triangleleft M$. Now $H = AA^b$ is a group, $Z = A \cap A^b \subseteq \mathbf{Z}(H)$ and H/Z is Abelian and thus H is nilpotent of class $\leqslant 2$. If $|A|$ is odd, then $|H|$ is odd and the exponent of H is equal to that of A.

Since B is Abelian, $A^b \cap B = (A \cap B)^b = A \cap B$ and hence $A \cap B = Z \cap B$. Also $H \cap B$ is Abelian and $Z \subseteq \mathbf{Z}(H)$ so $(H \cap B)Z$ is Abelian. Set

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$A^* = (H \cap B)Z$. Since $A^* \subseteq H$, its exponent divides that of H and (d) follows. Now $A < H \subseteq BA$ and so $H = A(H \cap B)$ and $H \cap B \nsubseteq A$. Hence $A^* \cap B \supseteq H \cap B > A \cap B$. Since $A \lhd M$, we clearly have $A^* \subseteq \mathbf{N}(A)$ and all that remains is to show (a). We have

$$| A^* | = | H \cap B | | Z |/| Z \cap B | = | H \cap B | | Z |/| A \cap B |.$$

However, from $H = AA^b = A(H \cap B)$, we obtain

$$| H | = | A |^2/| Z | = | A | | H \cap B |/| A \cap B |$$

and the result follows.

We mention an application to show how (d) can be used. Of course, the following can also be proved directly.

COROLLARY. *Let P be a p-group for $p \neq 2$ and let $B < P$ be Abelian. Suppose $x \in P$ has order p and $x \in \mathbf{C}[\Omega_1(B)]$. Then $[B, x] \subseteq \Omega_1(B)$.*

Proof. Let $A = \langle \Omega_1(B), x \rangle$, so that A is elementary Abelian. If B does not normalize A, then, by the theorem, there exists elementary Abelian A^* with $A^* \cap B > A \cap B$. This is impossible since $A \cap B = \Omega_1(B)$, and we conclude that $B \subseteq \mathbf{N}(A)$. Then $[B, x] \subseteq A \cap B = \Omega_1(B)$.

REFERENCE

1. D. GORENSTEIN, "Finite Groups." Harper and Row, New York, 1968.

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